Lower bounds

So far we have seen positive results: basic algorithmic techniques for fixed-parameter tractability.

What kind of negative results we have?

- Can we show that a problem (e.g., CLIQUE) is **not** FPT?
- Can we show that a problem (e.g., VERTEX COVER) has **no** algorithm with running time, say, $2^{o(k)} \cdot n^{O(1)}$?

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This would require showing that $\mathrm{P} \neq \mathrm{NP}$: if $\mathrm{P} = \mathrm{NP}$, then, e.g., k-Clique is polynomial-time solvable, hence FPT.

Can we give some evidence for negative results?

Classical complexity — reminder

NP:

- The class of all languages that can be recognized by a polynomial-time NTM.
- The class of all languages with a witness of polynomial size

Nondeterministic Turing Machine (NTM): single tape, finite alphabet, finite state, head can move left/right only one cell. In each step, the machine can branch into an arbitrary number of directions. Run is successful if at least one branch is successful.

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Polynomial-time reduction from problem P to problem Q: a function ϕ with the following properties:

- $\phi(x)$ is a yes-instance of $Q \iff x$ is a yes-instance of P,
- $\phi(x)$ can be computed in time $|x|^{O(1)}$.

Definition: Problem Q is NP-hard if any problem in NP can be reduced to Q.

If an NP-hard problem can be solved in polynomial time, then every problem in NP can be solved in polynomial time (i.e., $\mathsf{P} = \mathsf{NP}$).

Parameterized complexity

To build a complexity theory for parameterized problems, we need two concepts:

- An appropriate notion of reduction.
- An appropriate hypothesis.

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Fact: Graph G has an independent set $k \Leftrightarrow G$ has a vertex cover of size $n - k$.

Independent Set (G, k)	$\Rightarrow$	Vertex Cover $(G, n - k)$

- This is a correct polynomial-time reduction.
- However, Vertex Cover is FPT, but Independent Set is not known to be FPT.

Parameterized reductions

Definition

Parameterized reduction from problem A to problem B: a function ϕ with the following properties:

- $\phi(x)$ is a yes-instance of $B \iff x$ is a yes-instance of A,
- $\phi(x)$ can be computed in time $f(k) \cdot |x|^{O(1)}$, where k is the parameter of x,
- If k is the parameter of x and k' is the parameter of $\phi(x)$, then $k' \leq g(k)$ for some function g.

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Theorem

If there is a parameterized reduction from problem A to problem B and B is FPT, then A is also FPT.

Intuitively: Reduction $A \rightarrow B$ + algorithm for B gives and algorithm for A.

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Non-example: Transforming an Independent Set instance (G, k) into a Vertex Cover instance $(G, n - k)$ is **not** a parameterized reduction.

Example: Transforming an Independent Set instance (G, k) into a Clique instance $(\overline{G}, k)$ **is** a parameterized reduction.

Parameterized reductions

Theorem

If there is a parameterized reduction from problem A to problem B and B is FPT, then A is also FPT.

Proof: Suppose that

- the reduction has running time $f(k)n^{c_1}$,
- the reduction creates an instance with parameter at most $g(k)$, and
- B can be solved in time $h(k)n^{c_2}$.

Then running the reduction an solving the created instance of B gives an algorithm for A with running time

$$f(k)n^{c_1} + h(g(k)) \cdot (f(k)n^{c_1})^{c_2} \leq f'(k)n^{c_1 c_2}$$

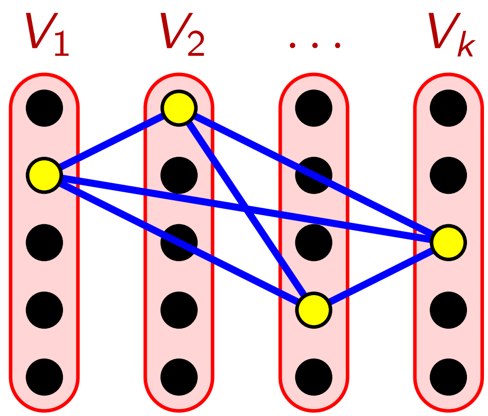
for some function f'.

Multicolored Clique

A useful variant of Clique:

Multicolored Clique: The vertices of the input graph G are colored with k colors and we have to find a clique containing one vertex from each color.

(or Partitioned Clique)



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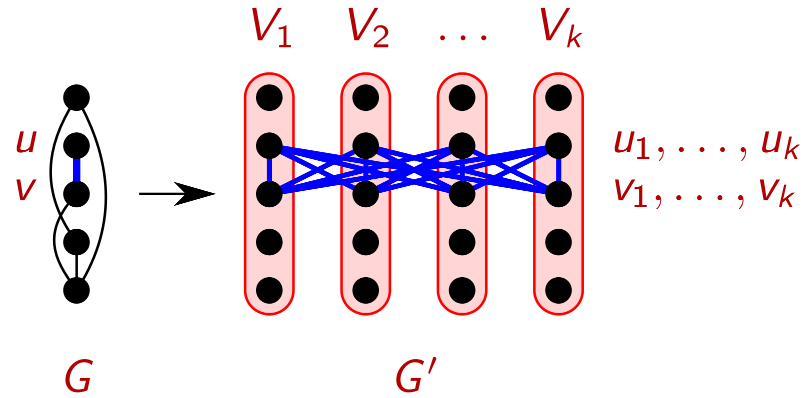
Create G' by replacing each vertex v with k vertices, one in each color class. If u and v are adjacent in the original graph, connect all copies of u with all copies of v.



k-clique in G $\iff$ multicolored k-clique in G'.

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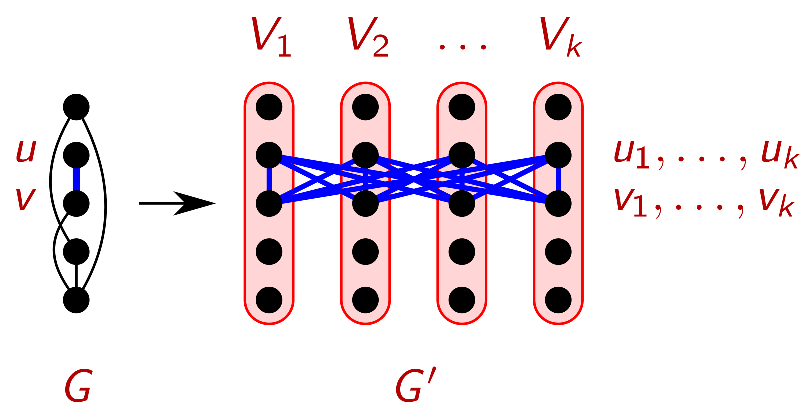


k-clique in G $\iff$ multicolored k-clique in G'.

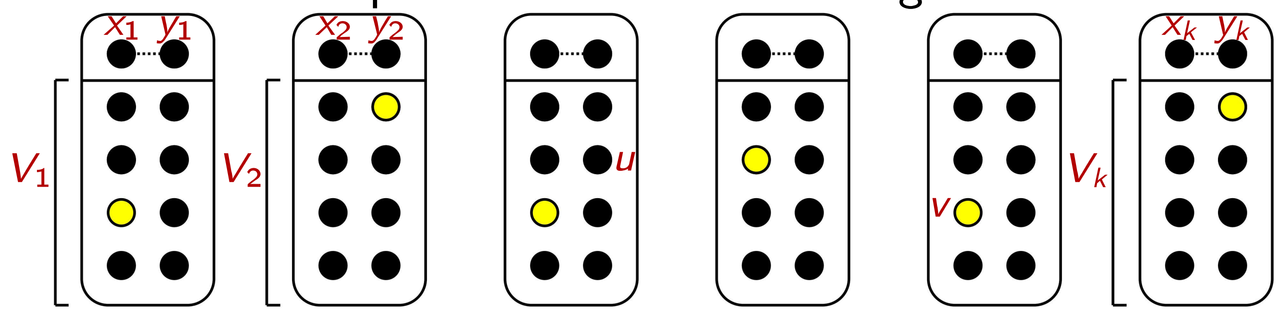
Similarly: reduction to Multicolored Independent Set.

Dominating Set

Theorem

There is a parameterized reduction from Multicolored Independent Set to Dominating Set.

Proof: Let G be a graph with color classes $V_1, \ldots, V_k$. We construct a graph H such that G has a multicolored k-clique iff H has a dominating set of size k.



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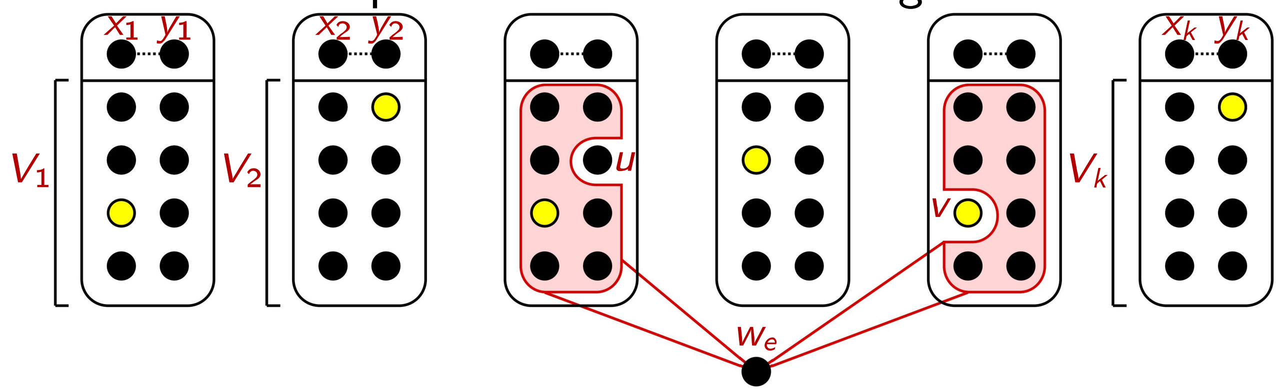

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- For every edge $e = uv$, an additional vertex w_e ensures that these selections describe an independent set.

Variants of Dominating Set

- Dominating Set: Given a graph, find k vertices that dominate every vertex.
- Red-Blue Dominating Set: Given a bipartite graph, find k vertices on the red side that dominate the blue side.
- Set Cover: Given a set system, find k sets whose union covers the universe.
- Hitting Set: Given a set system, find k elements that intersect every set in the system.

All of these problems are equivalent under parameterized reductions, hence at least as hard as Clique.

Basic hypotheses

It seems that parameterized complexity theory cannot be built on assuming $\mathsf{P} \neq \mathsf{NP}$ – we have to assume something stronger.

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k-CLIQUE cannot be solved in time $f(k) \cdot n^{O(1)}$.

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k-Step Halting Problem (is there a path of the given NTM that stops in k steps?) cannot be solved in time $f(k) \cdot n^{O(1)}$.

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n-variable 3SAT cannot be solved in time $2^{o(n)}$.

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Summary

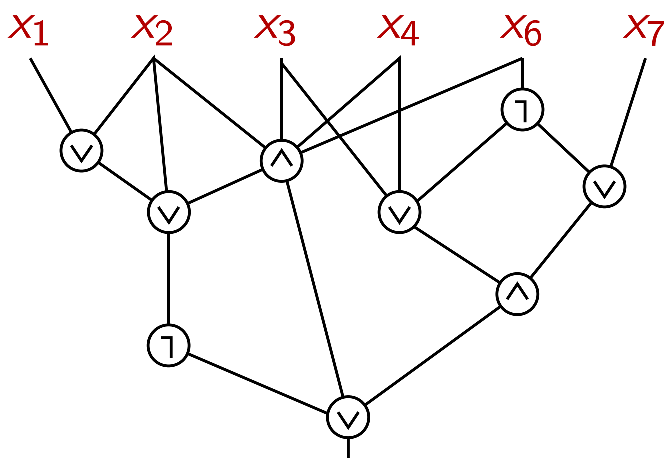
- INDEPENDENT SET and k-STEP HALTING PROBLEM can be reduced to each other $\Rightarrow$ Engineers' Hypothesis and Theorists' Hypothesis are equivalent!
- INDEPENDENT SET and k-STEP HALTING PROBLEM can be reduced to DOMINATING SET.

Summary

- Independent Set and k-Step Halting Problem can be reduced to each other $\Rightarrow$ Engineers' Hypothesis and Theorists' Hypothesis are equivalent!
- Independent Set and k-Step Halting Problem can be reduced to Dominating Set.
- Is there a parameterized reduction from Dominating Set to Independent Set?
- Probably not. Unlike in NP-completeness, where most problems are equivalent, here we have a hierarchy of hard problems.
  - Independent Set is W[1]-complete.
  - Dominating Set is W[2]-complete.
- Does not matter if we only care about whether a problem is FPT or not!

Boolean circuit

A **Boolean circuit** consists of input gates, negation gates, AND gates, OR gates, and a single output gate.



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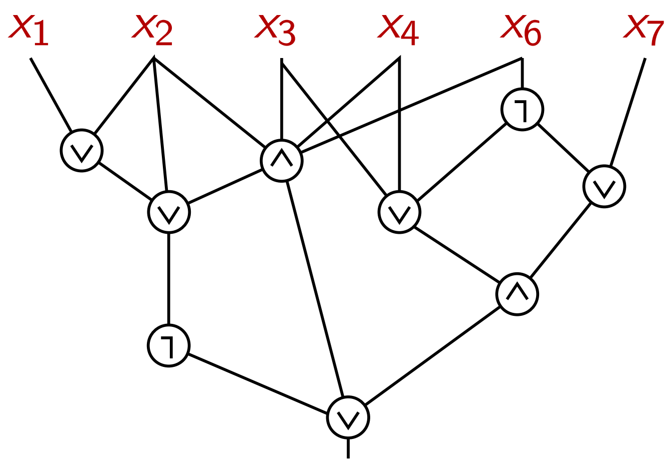


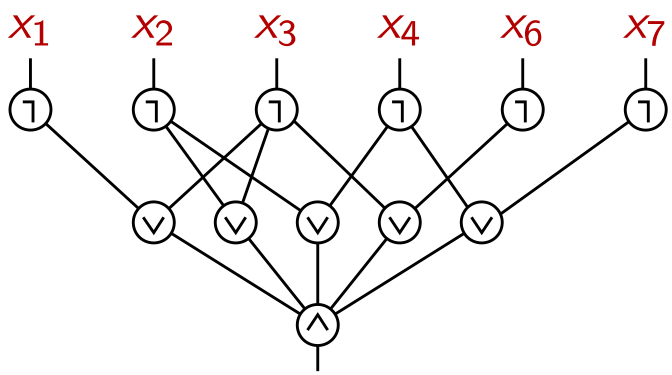
CIRCUIT SATISFIABILITY: Given a Boolean circuit C, decide if there is an assignment on the inputs of C making the output true.

Weight of an assignment: number of true values.

WEIGHTED CIRCUIT SATISFIABILITY: Given a Boolean circuit C and an integer k, decide if there is an assignment of weight k making the output true.

Weighted Circuit Satisfiability

Independent Set can be reduced to Weighted Circuit Satisfiability:



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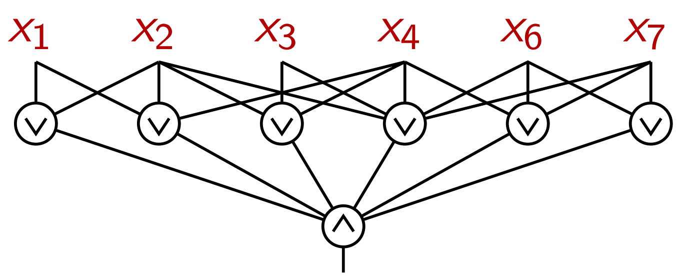

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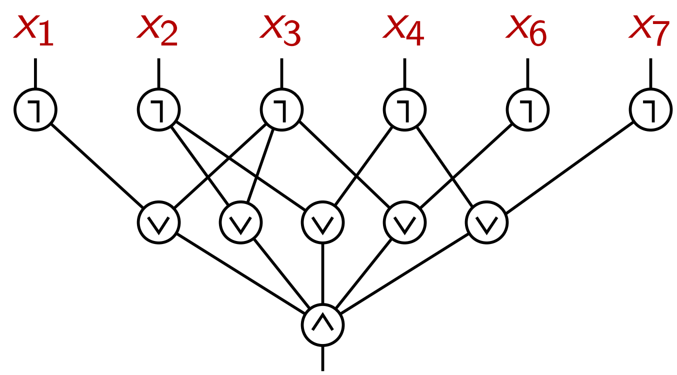

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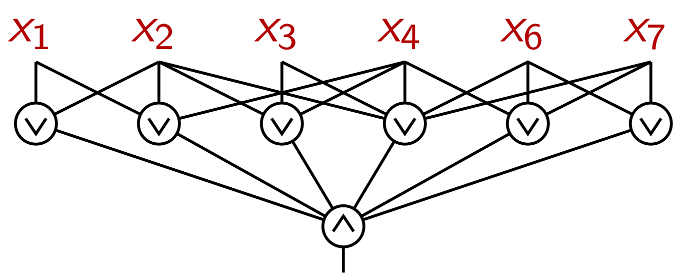


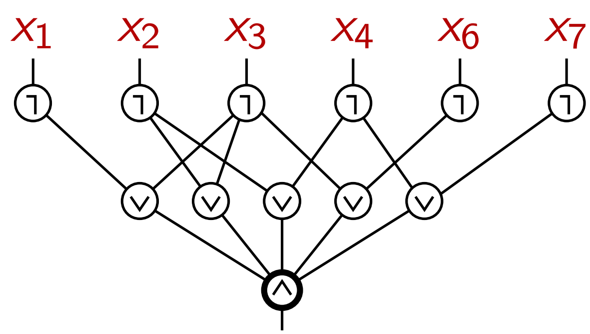
To express Dominating Set, we need more complicated circuits.

Depth and weft

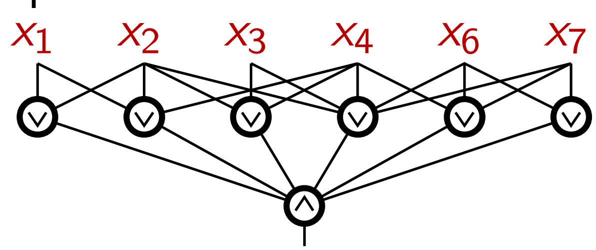
The **depth** of a circuit is the maximum length of a path from an input to the output.

A gate is **large** if it has more than 2 inputs. The **weft** of a circuit is the maximum number of large gates on a path from an input to the output.

INDEPENDENT SET: weft 1, depth 3



DOMINATING SET: weft 2, depth 2



The W-hierarchy

Let $C[t, d]$ be the set of all circuits having weft at most t and depth at most d.

Definition

A problem P is in the class W[t] if there is a constant d and a parameterized reduction from P to WEIGHTED CIRCUIT SATISFIABILITY of $C[t, d]$.

We have seen that INDEPENDENT SET is in W[1] and DOMINATING SET is in W[2].

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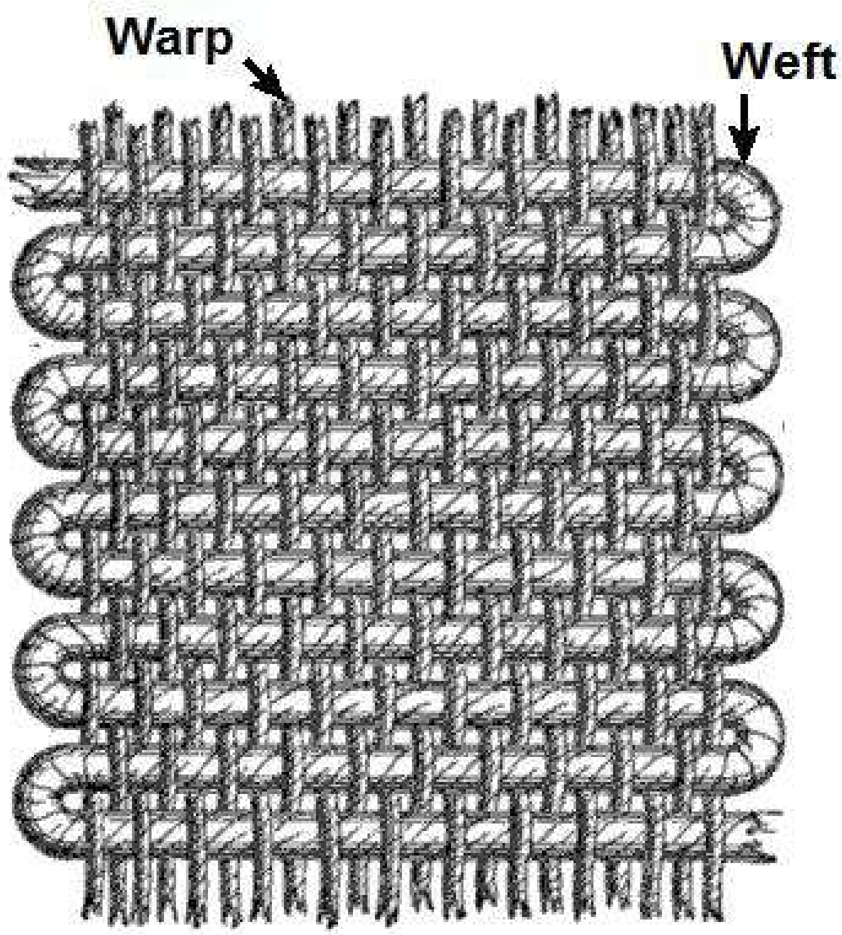
Fact: INDEPENDENT SET is W[1]-complete.
Fact: DOMINATING SET is W[2]-complete.

If any W[1]-complete problem is FPT, then FPT $=$ W[1] and **every** problem in W[1] is FPT.

If any W[2]-complete problem is in W[1], then W[1] $=$ W[2].

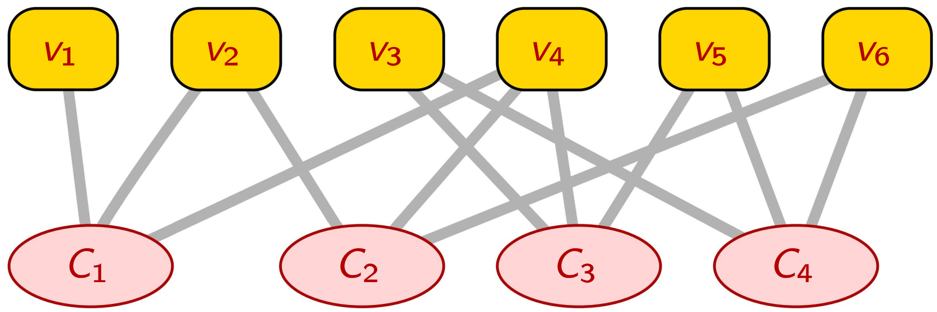
$\Rightarrow$ If there is a parameterized reduction from DOMINATING SET to INDEPENDENT SET, then W[1] $=$ W[2].

Weft



Parameterized reductions

Typical NP-hardness proofs: reduction from e.g., Clique or 3SAT, representing each vertex/edge/variable/clause with a gadget.



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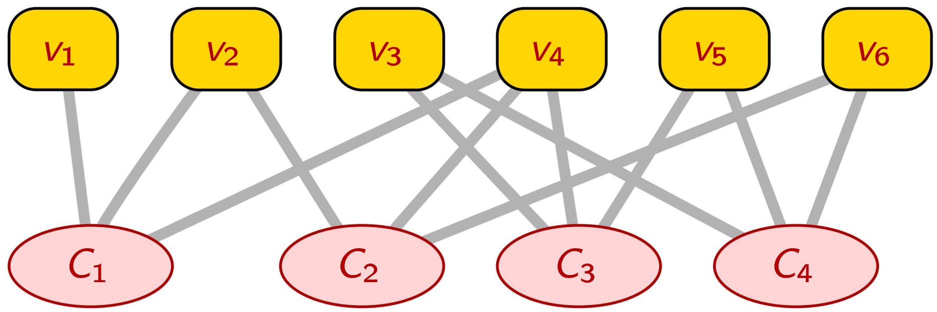

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Types of parameterized reductions:

- Reductions keeping the structure of the graph.
  - CLIQUE ⇒ INDEPENDENT SET
- Reductions with vertex representations.
  - MULTICOLORED INDEPENDENT SET ⇒ DOMINATING SET
- Reductions with vertex and edge representations.

Odd Set

Odd Set: Given a set system $\mathcal{F}$ over a universe U and an integer k, find a set S of at most k elements such that $|S \cap F|$ is odd for every $F \in \mathcal{F}$.

Theorem

Odd Set is W[1]-hard parameterized by k.

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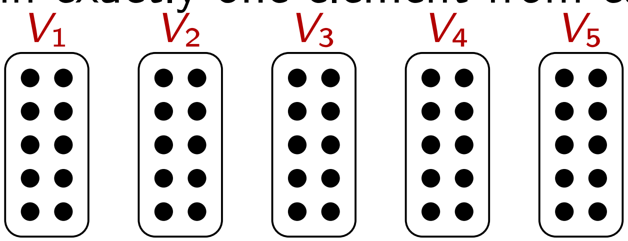
First try: Reduction from Multicolored Independent Set. Let $U = V_1 \cup \dots V_k$ and introduce each set V_i into $\mathcal{F}$.

$\Rightarrow$ The solution has to contain exactly one element from each V_i.



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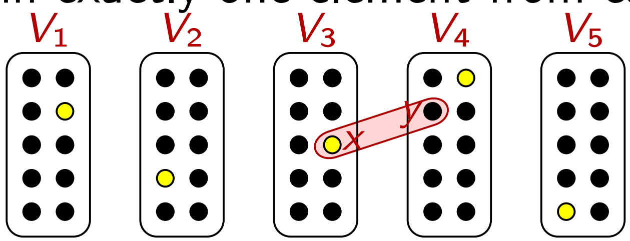

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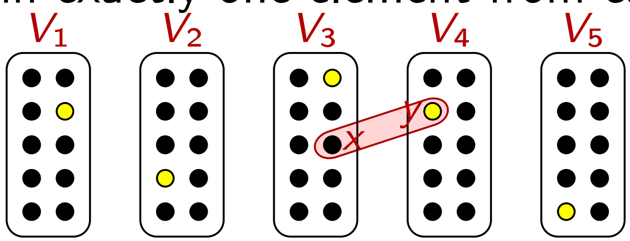

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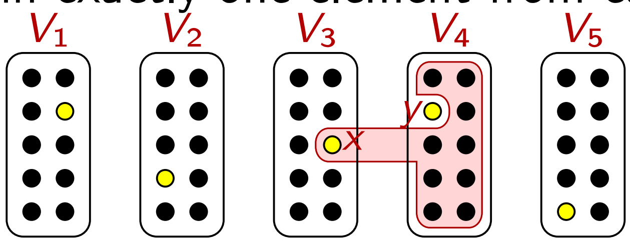

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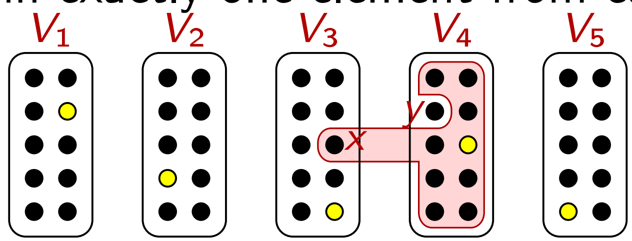

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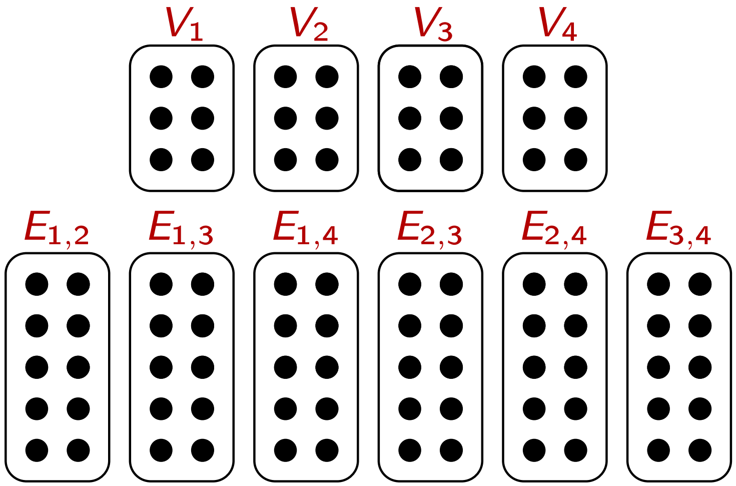

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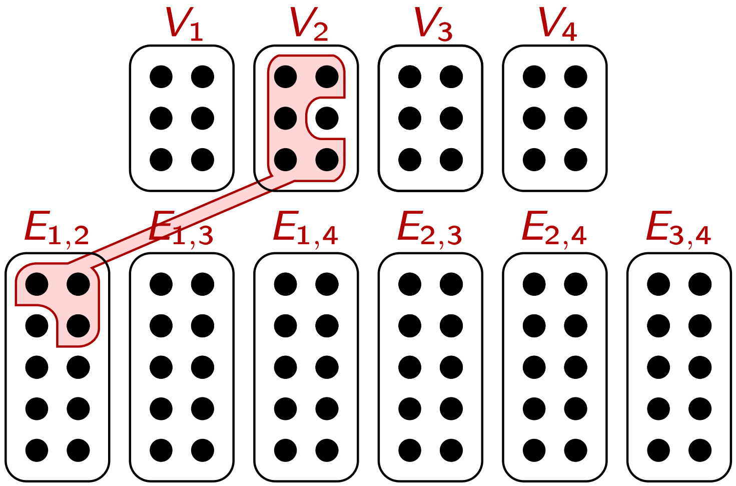

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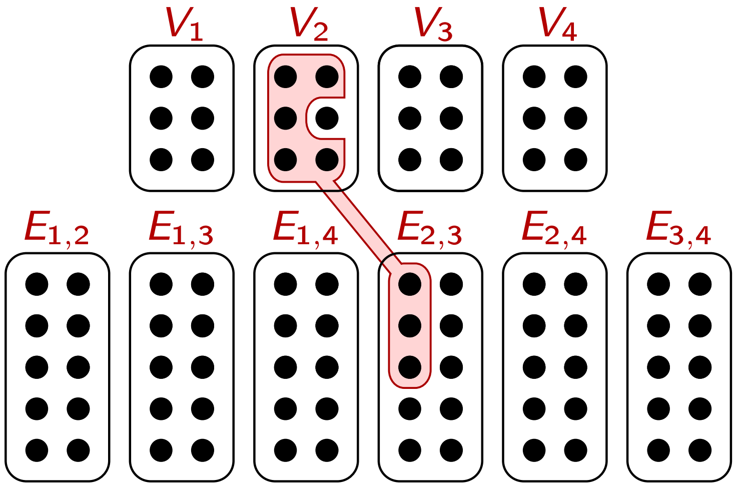

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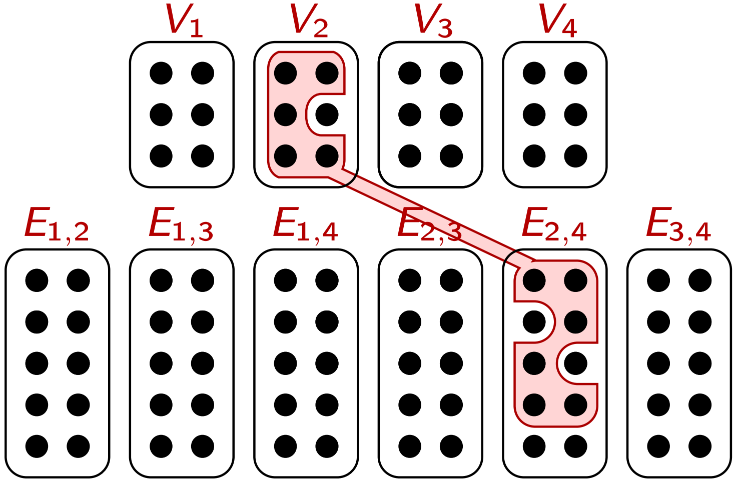

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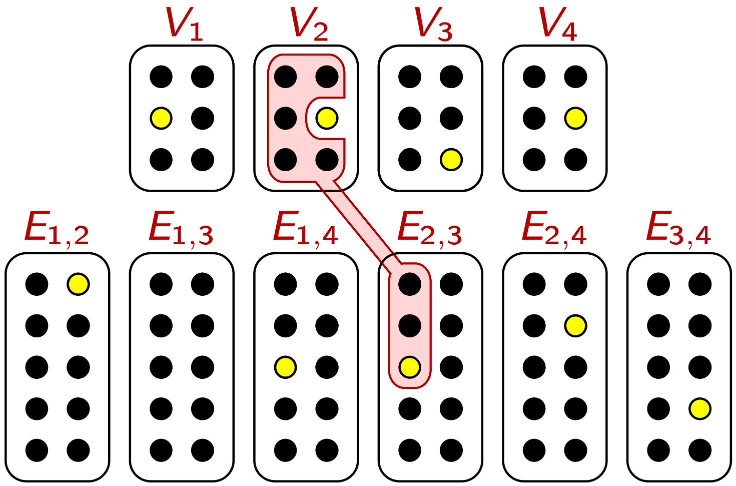
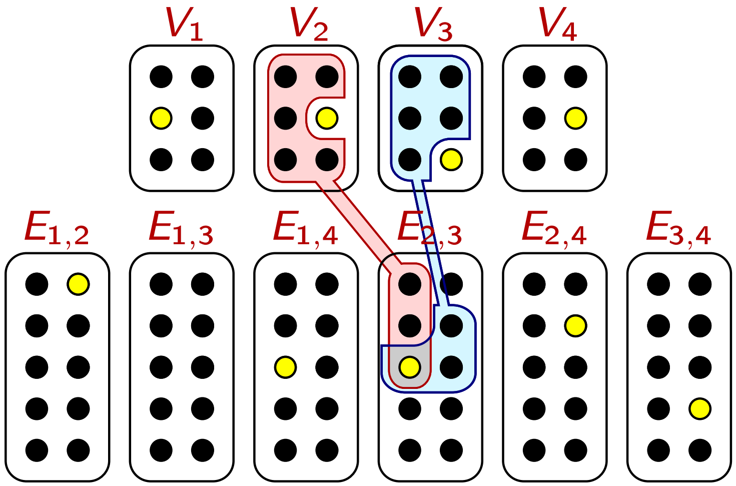
  $(V_i \setminus \{v\}) \cup \{\text{every edge from } E_{i,x} \text{ with endpoint } v\}$
  $(V_i \setminus \{v\}) \cup \{\text{every edge from } E_{x,i} \text{ with endpoint } v\}$
- $v \in V_i$ selected $\iff$ edges with endpoint v are selected from $E_{i,x}$ and $E_{x,i}$



Odd Set

Reduction from Multicolored Clique.

- For every $v \in V_i$ and $x \neq i$, we introduce the sets:
  $(V_i \setminus \{v\}) \cup \{$every edge from $E_{i,x}$ with endpoint $v\}$
  $(V_i \setminus \{v\}) \cup \{$every edge from $E_{x,i}$ with endpoint $v\}$
- $v \in V_i$ selected $\iff$ edges with endpoint v are selected from $E_{i,x}$ and $E_{x,i}$
- $v_i \in V_i$ selected, $v_j \in V_j$ selected $\iff$ edge $v_i v_j$ is selected in $E_{i,x}$



Vertex and edge representation

Key idea

- Represent the vertices of the clique by k gadgets.
- Represent the edges of the clique by $\binom{k}{2}$ gadgets.
- Connect edge gadget $E_{i,j}$ to vertex gadgets V_i and V_j such that if $E_{i,j}$ represents the edge between $x \in V_i$ and $y \in V_j$, then it **forces** V_i to x and V_j to y.

Variants of HITTING SET

The following problems are W[1]-hard, with very similar proofs:

- ODD SET
- EXACT ODD SET (find a set of size exactly k ...)
- EXACT EVEN SET
- UNIQUE HITTING SET
  (at most k elements that hit each set exactly once)
- EXACT UNIQUE HITTING SET
  (exactly k elements that hit each set exactly once)

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A problem that is also W[1]-hard, but requires very different techniques:

- EVEN SET: Given a set system $\mathcal{F}$ and an integer k, find a **nonempty** set S of at most k elements such $|F \cap S|$ is even for every $F \in \mathcal{F}$.

Summary

- By parameterized reductions, we can show that lots of parameterized problems are at least as hard as CLIQUE, hence unlikely to be fixed-parameter tractable.
- Connection with Turing machines gives some supporting evidence for hardness (only of theoretical interest).
- The W-hierarchy classifies the problems according to hardness (only of theoretical interest).
- Important trick in W[1]-hardness proofs: vertex and edge representations.